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Electromagnetism and gauge theory on the permutation group S_3

Shahn Majid*, E. Raineri

School of Mathematical Sciences, Queen Mary and Westfield College, University of London, Mile End Road, London E1 4NS, UK

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Abstract

Using noncommutative geometry we do $U(1)$ gauge theory on the permutation group S_3. Unlike usual lattice gauge theories the use of a non-Abelian group here as spacetime corresponds to a background Riemannian curvature. In this background we solve spin 0, $\frac{1}{2}$ and spin 1 equations of motion, including the spin 1 or 'photon' case in the presence of sources, i.e. a theory of classical electromagnetism. Moreover, we solve the $U(1)$ Yang–Mills theory (this differs from the $U(1)$ Maxwell theory in noncommutative geometry), including the moduli space of flat connections. We show that the Yang–Mills action has a simple form in terms of Wilson loops in the permutation group, and we discuss aspects of the quantum theory.

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1. Introduction

As an attempt to make quantum theory computable it is common to consider its formulation on a flat lattice $\mathbb{Z}^n$ in place of spacetime $\mathbb{R}^n$. On the other hand, using modern methods of noncommutative geometry it is possible to formulate such constructions more 'geometrically' in terms of a noncommutative exterior algebra of differential forms and a Cartan calculus. In lattice approximations, the finite differences are indeed intrinsically noncommutative in the sense that they should be formulated better as bimodules over

* Corresponding author. Tel.: +44-1-223-337900; fax: +44-1-223-337-918.
E-mail address: s.majid@qmul.ac.uk (S. Majid).

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functions: the product of a function and a finite differential is naturally given by the value of the function either at the start-point or the end-point of the differential, and the two are different. Hence functions and 1-forms obey $f\,\mathrm{d}x \neq (\mathrm{d}x)f$, which means that such a more general noncommutative geometry is the natural way to do lattice theory.

In this paper we want to go much beyond this initial observation. In fact such methods of noncommutative geometry apply equally well for any Hopf algebra and hence in particular for any finite group G. This offers the possibility for the first time of a natural 'geometric' lattice approximation by non-Abelian finite groups rather than by a $\mathbb{Z}^n$ or $(\mathbb{Z}_m)^n$ lattice. The Abelian case is also interesting in noncommutative geometry, e.g. [1] or more recently [2]. However, the noncommutative theory comes into its own when we seek to model a space or spacetime with spherical or other topology. In particular, it has been shown recently in [3] that just as cyclic groups $\mathbb{Z}_m$ approximate tori, permutation groups such as S_3 (permutations on three elements) are more like compact semisimple Lie groups. It was shown that S_3 has a natural noncommutative Riemannian structure with Ricci curvature essentially proportional to the metric and translation-invariant (like a classical sphere S^3). The curvature originates in the non-Abelianess of the group S_3.

Other metrics and connections also exist and in principle one could proceed to gravity and quantum gravity on S_3 using these methods. Before attempting such a project one should consider the simpler problem of spins 0, $\frac{1}{2}$, 1 fields moving in the natural Killing-form metric Riemannian background. This is what we do in the present paper. In the natural 3-bein coordinates, the Killing metric just turns out to be [3] the Euclidean δ_{ab}. Using this we then define the Hodge $\star$ operator and hence such things as the Maxwell and Yang–Mills Lagrangians $(\mathrm{d}F)^* \wedge \star F$. The classical theory particularly of 'electromagnetism' explores in effect the classical noncommutative geometry of S_3. We compute the quantum deRham cohomology (it is nontrivial) and linear wave equations, etc. in Section 2. We also obtain point sources and dipole sources for the Maxwell field. We explain the required Coulomb gauge fixing and more or less completely treat the linear system.

In Section 3 we look at the nonlinear $U(1)$ Yang–Mills theory with $F = \mathrm{d}A + A \wedge A$ (this is not the same as the linearised Maxwell theory due to the non(super)commutativity of the differential forms). We find the moduli space of flat connections, which turns out to be nontrivial. We also look for instantons but show that none exist obeying the required reality conditions. Finally, we show that the Lagrangian in the Yang–Mills case has a nice description in terms of a real 'kinetic' term and Wilson loops around elementary plaquettes

$$L = \lambda_u^2 \partial^u \lambda_v^2 + \lambda_u^2 \lambda_v^2 - W_u(A) + \text{cyclic rotations},$$

where u, v, w are the transpositions of S_3 and label the tangent space at each point $x \in S_3$. λ_u, etc. are real positive fields built from A (essentially we use polar coordinates for the values of A) and

$$W_u(A)(x) = (1 + A^u(x))(1 + A^v(xu))(1 + A^u(xuv))(1 + A^w(xw))$$

is the holonomy around a small square at x with sides u, v, u, w in the group. It is remarkable that we do not put this in by hand as some kind of approximation (as one does in conventional lattice theory), it is literally what we obtain for $F^* \wedge \star F$ using the noncommutative differential geometry on S_3 and the Riemannian structure from [3]. This extends what has been observed for lattice $\mathbb{R}^n$, e.g. in [1]. Our Riemannian geometry approach works equally for essentially all quantum groups and many other systems, though we do not discuss them here.

We conclude in Section 4 with some remarks about the quantum theory. There being only six points in S_3, functional integrals over our fields become multiple usual integrals. We formulate the required actions based on minimal coupling and also explain how to compute the partition function and expectation values of Wilson loops $\langle W_u(A)(x)\rangle$. All of this should be viewed as a warm up to functional integrals over metrics and their connections, i.e. quantum gravity where our finite method should be particularly useful. An introduction to the framework of gravity in our approach (which plays only a background role) is in [4].

1.1. Preliminaries

Here we recall very briefly the formalism of noncommutative differential geometry for finite groups G. This Hopf algebra approach to noncommutative geometry coming out of quantum groups should not be confused with the approaches to noncommutative geometry of Connes [5], though the treatment of 1-forms as bimodules is common to both, and there are some models where the two methods begin nontrivially to 'converge' [2].

In the quantum groups approach, we work with the algebra $\mathbb{C}[G]$ of functions on G. We do *not* consider derivations as vector fields (this does not work here) but rather we define $\Omega(G)$ the exterior algebra of forms as a $\mathbb{Z}_2$-graded algebra with d a super-derivation and $\mathrm{d}^2 = 0$. Using the construction of [6] this is specified in a bicovariant manner entirely by an Ad-stable subset $\mathcal{C}$ not containing the group identity e. The 1-forms have a basis $\{e_a : a \in \mathcal{C}\}$ over $\mathbb{C}[G]$, bimodule structure and d on functions

$$\Omega^1 = \langle e_a\rangle, \quad e_a f = R_a(f)e_a, \qquad \mathrm{d}f = \sum_a (\partial^a f)e_a, \quad \partial^a = R_a - \mathrm{id},$$

where $R_a(f)(x) = f(xa)$ for all $x \in G$ and $a \in \mathcal{C}$. The elements of $\mathcal{C}$ are the 'allowed directions'. The partial derivatives defined here obey a braided [7] Leibniz rule

$$\partial^a(fg) = \partial^a(f)g + R_a(f)\partial^a(g), \quad \forall f, g \in \mathbb{C}[G].$$

The higher forms are a certain quotient of the tensor power of 1-forms where we set to 0 those 'symmetric' combinations invariant under a braided-symmetrization operator defined by a certain braiding Ψ. The d is extended through the Maurer–Cartan relation

$$\mathrm{d}e_a = e_a \wedge \theta + \theta \wedge e_a, \quad \theta \equiv \sum_a e_a$$

and the graded Leibniz rule. From this one also finds that

$$\mathrm{d}\alpha = [\theta, \alpha\}, \quad \forall \alpha \in \Omega(G)$$

using the graded anti-commutator. Also

$$e_{a_1}\wedge\cdots\wedge e_{a_m} f = R_{a_1\cdots a_m}(f) e_{a_1}\wedge\cdots\wedge e_{a_m},$$

where the product $a_1\cdots a_m$ defines a natural G-valued degree on $\Omega(G)$. Further details of the set-up including the required quotient at degree 2 for general G are in [3].

For S_3 we take generators and relations, and conjugacy class

$$u^2=v^2=e,\qquad uvu=vuv\equiv w,\qquad \mathcal{C}=\{u,v,w\}.$$

So $\Omega^1=\langle e_u,e_v,e_w\rangle$. Because every element of $\mathcal{C}$ has order 2, we have

$$R_a\partial^a=-\partial^a,\qquad (\partial^a)^2=-2\partial^a$$

for all $a=u,v,w$. It is also easy to see that degree 2 relations

$$e_u\wedge e_v+e_v\wedge e_w+e_w\wedge e_u=0,$$
$$e_v\wedge e_u+e_w\wedge e_v+e_u\wedge e_w=0,\qquad e_u\wedge e_u=e_v\wedge e_v=e_w\wedge e_w=0$$

hold. It is well-known that these are in fact the only relations in degree 2 in the Woronowicz construction (an actual proof is in [3]). Hence Ω^1 is 3D (3-dimensional) while Ω^2 is 4D. As a basis of the latter we choose (for concreteness)

$$\Omega^2=\langle e_u\wedge e_v, e_v\wedge e_u, e_v\wedge e_w, e_w\wedge e_v\rangle.$$

Next, one easily computes the consequences of the degree 2 relations in higher degree, which we call the 'quadratic prolongation' of Ω^1. It has been used for S_3 in [8] and recently, e.g. in [9,3] and one has

$$e_u\wedge e_v\wedge e_w=e_w\wedge e_v\wedge e_u=-e_w\wedge e_u\wedge e_w=-e_u\wedge e_w\wedge e_u$$

and the two cyclic rotations $u\to v\to w\to u$ of these relations. Hence there are three independent 3-forms

$$\Omega^3=\langle e_w\wedge e_u\wedge e_v, e_u\wedge e_v\wedge e_w, e_v\wedge e_w\wedge e_u\rangle$$

in the quadratic prolongation. Similarly there is one independent 4-form with

$$\mathrm{Top}\equiv e_u\wedge e_v\wedge e_u\wedge e_w=e_v\wedge e_u\wedge e_v\wedge e_w$$
$$=-e_w\wedge e_u\wedge e_v\wedge e_u=-e_w\wedge e_v\wedge e_u\wedge e_v$$

and equal to the two cyclic rotations of these equations (Top is invariant). Any expression of the form $e_a\wedge e_b\wedge e_a\wedge e_b$ is 0 as is any expression with a repetition in the outer (or inner) two positions. It is easy to see that the basic 2-forms mutually commute and that Top has trivial total G-degree.

It turns out that *the quadratic prolongation in this case is exactly* $\Omega(S_3)$, i.e. there are no further relations imposed by the braided-antisymmetrization process in higher degree in this case. This is not expected to hold in general and we have not seen an actual proof of this fact for S_3, therefore, we include it now for completeness.

Lemma 1.1. *There are no further relations from Woronowicz's braided-antisymmetrization procedure, i.e.* $\Omega(S_3)$ *has dimensions* 1:3:4:3:1 *as for the quadratic prolongation*.

Proof. According to [6] we have to compute the dimension of the kernel of

$$A_3 = \mathrm{id} - \Psi_{12} - \Psi_{23} + \Psi_{12}\Psi_{23} + \Psi_{23}\Psi_{12} - \Psi_{12}\Psi_{23}\Psi_{12}$$

acting on $\Omega^1 \otimes \Omega^1 \otimes \Omega^1$ (tensor over $\mathbb{C}[S_3]$). Here the braiding is $\Psi(e_a \otimes e_b) = e_{aba^{-1}} \otimes e_a$. To find the dimension of the kernel, one first checks that $A_3(e_a \otimes e_b \otimes e_c) = 0$ as soon as $a = b$ or $b = c$. The null space of A_3 spanned by these vectors has a complement $V = \oplus_{c \in \mathcal{C}} V_c$, where for $c \in \mathcal{C}$ and (a, b, c) a cyclic permutation of (u, v, w), V_c has basis

$$\{e_a \otimes e_b \otimes e_a, e_b \otimes e_a \otimes e_b, e_b \otimes e_c \otimes e_a, e_a \otimes e_c \otimes e_b\}.$$

One finds that each V_a is preserved by A_3, and A_3 is given by this 4×4 matrix (in the chosen basis)

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{pmatrix},$$

which is diagonalisable with eigenvalues $(0, 0, 0, 4)$. Therefore, $\dim A_3(V_c) = 1$ for all $c \in \mathcal{C}$, and $\dim(A_3(\Omega^1)^{\otimes 3}) = \sum_{a \in \mathcal{C}} 1 = 3$. Hence Ω^3 which is defined as the tensor cube of Ω^1 modulo $\ker A_3$ is 3D, which is the same as the quadratic prolongation, so that there are no further relations in degree 3. Notice that A_3 is not a projector, but $(1/4)A_3$ is.

In degree 4, we check that Top is not in the kernel of A_4 (defined similarly) and hence that there is no further quotient in degree 4. □

Next it is obvious in the presence of a Top form that one can define $e_a \wedge e_b \wedge e_c \wedge e_d = \epsilon_{abcd}\mathrm{Top}$ for all $a, b, c \in \mathcal{C}$. This is not yet enough to proceed in to a Hodge $\star$ operator because for that one needs a Riemannian metric η_{ab}. However, this is precisely what comes out of the theory of Riemannian structures on finite groups and quantum groups [3] from the 'braided Killing form' of the tangent space braided-Lie algebra. For S_3 (in a suitable normalisation) it just turns out to be $\eta_{ab} = \delta_{ab}$, the Euclidean metric in the natural 3-bein coordinates provided by the e_a themselves. Using this we now introduce the Hodge $\star$ operator

$$\begin{aligned} \star(e_{a_1} \wedge \cdots \wedge e_{a_m}) &= d_m^{-1} \epsilon_{a_1 \cdots a_m b_{m+1} \cdots b_n} \eta^{b_{m+1}c_{m+1}} \cdots \eta^{b_n c_n} e_{c_n} \wedge \cdots \wedge e_{c_{m+1}} \\ &= d_m^{-1} \epsilon_{a_1 \cdots a_n} e_{a_n} \wedge \cdots \wedge e_{a_{m+1}} \end{aligned}$$

for some normalisation constants d_m. The ordering of indices is determined so that the total G-degree (as above) is preserved by $\star$ (here every element of $\mathcal{C}$ has order 2 or we would need inverses on the right-hand side). In our case we take

$$d_0 = 1, \quad d_1 = 2, \quad d_2 = \sqrt{3}, \quad d_3 = 2, \quad d_4 = 1.$$

In this way one finds:

Proposition 1.2. *The natural Hodge $\star$ operator on $\Omega(S_3)$ is*

$$\begin{aligned}
&\star(1) = \mathrm{Top}, \quad \star(e_u) = 2e_w \wedge e_u \wedge e_v, \quad \star(e_v) = 2e_u \wedge e_v \wedge e_w,\\
&\star(e_w) = 2e_v \wedge e_w \wedge e_u, \quad \star(e_u \wedge e_v) = -3^{-1/2}(e_u \wedge e_v + 2e_v \wedge e_w),\\
&\star(e_v \wedge e_w) = 3^{-1/2}(e_v \wedge e_w + 2e_u \wedge e_v), \quad \star(e_v \wedge e_u) = 3^{-1/2}(e_v \wedge e_u + 2e_w \wedge e_v),\\
&\star(e_w \wedge e_v) = -3^{-1/2}(e_w \wedge e_v + 2e_v \wedge e_u), \quad \star(e_w \wedge e_u \wedge e_v) = -\tfrac{1}{2}e_u,\\
&\star(e_u \wedge e_v \wedge e_w) = -\tfrac{1}{2}e_v, \quad \star(e_v \wedge e_w \wedge e_u) = -\tfrac{1}{2}e_w,\\
&\star\mathrm{Top} = -1
\end{aligned}$$

extended as a bimodule map. It obeys $\star^2 = -\mathrm{id}$.

Proof. By its construction it is clear that $\star$ has square -1 and preserves the G-degree. The latter means that if we define $\star(fe_{a_1} \wedge \cdots \wedge e_{a_m}) = f\star(e_{a_1} \wedge \cdots \wedge e_{a_m})$ for any function f then also $\star(e_{a_1} \wedge \cdots \wedge e_{a_m} f) = \star(R_{a_1 \cdots a_m}(f) e_{a_1} \wedge \cdots \wedge e_{a_m}) = R_{a_1 \cdots a_m}(f)\star(e_{a_1} \wedge \cdots \wedge e_{a_m}) = \star(e_{a_1} \wedge \cdots \wedge e_{a_m}) f$ as required. Note also that since Top is cyclically invariant there is also a cyclic invariance of $\star$. □

Also associated to this metric is a Riemannian covariant derivative, spin connection and Dirac operator. We will need the latter (coupled to a further $U(1)$ gauge field) in later sections. However, for spins 0, 1, one may proceed with only the Hodge $\star$ as above. As far as we know this Riemannian and Hodge structure goes beyond what has been considered before. Finally, whereas the above results hold (with different normalisations) over any field of characteristic 0, we also impose a complex $*$-algebra structure when we work over $\mathbb{C}$. Thus, we define

$$e_a^* = e_a, \qquad \mathrm{d}(\alpha^*) = (-1)^{|\alpha|+1}(\mathrm{d}\alpha)^*$$

and one may check that $\Omega(G)$ becomes a differential graded $*$-algebra. This should not be confused with the Hodge operator above.

2. Wave equations on S_3

In this section we write down Lagrangians and solve the associated linear wave equations for different spins. The spin 1 case means here 'Maxwell theory' or 1-forms modulo exact. This is a linearised version of the noncommutative $U(1)$ gauge theory in Section 3.

2.1. Spin 0

We consider a scalar field $\phi \in \mathbb{C}[S_3]$. From the definitions

$$(\mathrm{d}\phi)^* = e_a^* \overline{\partial^a \phi} = e_a \partial^a \bar{\phi} = R_a(\partial^a \bar{\phi}) e_a = -\partial^a \bar{\phi} e_a = -\mathrm{d}\bar{\phi}$$

as it should, and also note that

$$e_a \wedge \star(e_b) = 2\delta_{a,b}\,\mathrm{Top}.$$

Hence

$$L\,\mathrm{Top} \equiv -\frac{1}{2}(\mathrm{d}\phi)^* \wedge \bigstar(\mathrm{d}\phi) = \frac{1}{2}\sum_{a,b}(\partial^a\bar{\phi})e_a(\partial^b\phi)\bigstar(e_b) = \sum_a(\partial^a\bar{\phi})R_a(\partial^a\phi)\,\mathrm{Top}$$

gives the Lagranian density as

$$L = -\sum_a \partial^a\bar{\phi}\partial^a\phi$$

for scalar fields. Using the braided-Leibniz rule this is up to a total derivative

$$L = \sum_a(R_a\bar{\phi})(\partial^a)^2\phi = -\sum_a R_a(\bar{\phi}(\partial^a)^2\phi) = -\sum_a \bar{\phi}(\partial^a)^2\phi.$$

Hence the wave operator on spin 0 is

$$\Box = -\sum_a \partial^a\partial^a = \sum_a 2\partial^a.$$

It is easy to solve this. On a group manifold we would expect 'plane waves' associated to irreducible representations.

Proposition 2.1. *The only zero mode of $\Box$ is the constant function. In addition there is one mode of mass $2\sqrt{3}$ given by the sign representation, and four modes of mass $\sqrt{6}$ given by the matrix elements of the 2D representation of S_3.*

Proof. In our case S_3 has a trivial representation, which gives $\phi = 1$ with 'mass' 0. Then it has the sign representation which gives

$$\phi(x) = \mathrm{sign}(x) \equiv (-1)^{l(x)}, \qquad \Box\phi(x) = 2\sum_a((-1)^{l(xa)} - (-1)^{l(x)}) = -12\phi(x)$$

with 'mass' $2\sqrt{3}$ (here $l(x)$ is the length of the permutation or the number of u, v in its reduced expression). Finally, it has a 2×2 matrix representation

$$\rho(u) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \rho(v) = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

and each matrix element (for each $i, j = 1, 2$ fixed)

$$\phi_{ij}(x) = \rho(x)^i{}_j$$

is a 'mass' $\sqrt{6}$ since

$$\Box\phi_{ij}(x) = -6\phi_{ij}(x) + 2\sum_a\sum_k \rho(x)^i{}_k\rho(a)^k{}_j = -6\phi_{ij}(x)$$

as $\rho(u) + \rho(v) + \rho(w) = 0$. These four waves are linearly independent because the representation is irreducible. Since $\Box$ is a 6×6 matrix we have completely diagonalised it, i.e. its eigenvalues correspond to allowed masses $0, 2\sqrt{3}, \sqrt{6}$ with multiplicities $1, 1, 4$. $\Box$

Moreover, every function on S_3 has a unique decomposition of the form

$$\phi = p_0 + p_1 \mathrm{sign} + p_{ij}\phi_{ij}$$

for some numbers p_0, p_1, p_{ij} (real if we demand $\bar{\phi} = \phi$), i.e. a sum of our six waves. Associated to this decompositon is a projection of any function to its component waves (or non-Abelian Fourier transform). It is also worth noting that $\square$ is hermitian with respect to the usual L^2 inner product on S_3 and bicovariant hence its eigenspace decomposition must exist and be a decomposition into $S_3 \times S_3$ modules (similarly for any group G). In the S_3 case at least it is precisely the Peter–Weyl decomposition obtained in a new way.

We note that there is another useful construction of the projection to the mass $\sqrt{6}$ part, namely let ϕ_0 be any function and consider

$$\phi = 2\phi_0 - R_{uv}\phi_0 - R_{vu}\phi_0.$$

Then

$$\square\phi = -6\phi + 2\sum_a R_a(\phi) = -6\phi + 2\sum_a (2R_a\phi_0 - R_{auv}\phi_0 - R_{avu}\phi_0) = -6\phi$$

so ϕ is a solution of mass $\sqrt{6}$. One should divide by 3 for an actual projection of course. One may similarly project onto the other waves.

2.2. *Spin $\frac{1}{2}$*

For uncharged spin $\frac{1}{2}$ we use the 'curved space' Dirac operator introduced in [3]. There, the 'gamma-matrices' are given explicitly by

$$\gamma_u = \frac{1}{3}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad \gamma_v = \frac{1}{3}\begin{pmatrix} 0 & 0 \\ -1 & -2 \end{pmatrix}, \qquad \gamma_w = \frac{1}{3}\begin{pmatrix} -2 & -1 \\ 0 & 0 \end{pmatrix}$$

and obey

$$\gamma_a\gamma_b + \gamma_b\gamma_a + \frac{2}{3}(\gamma_a + \gamma_b) = \frac{1}{3}(\delta_{ab} - 1), \qquad \sum_a \gamma_a = -1. \tag{1}$$

There is a natural spin connection corresponding to the Killing-form metric on S_3 and including this, one has [3]

$$\begin{aligned} \not{D} = \partial^a\gamma_a - 1 &= \frac{1}{3}\begin{pmatrix} -\partial^u - 2\partial^w - 3 & \partial^u - \partial^w \\ \partial^u - \partial^v & -\partial^u - 2\partial^v - 3 \end{pmatrix} \\ &= \frac{1}{3}\begin{pmatrix} -R_u - 2R_w & R_u - R_w \\ R_u - R_v & -R_u - 2R_v \end{pmatrix}. \end{aligned}$$

It acts on 2-vector valued functions (spinors) on S_3. We note that if we let $\gamma = \mathrm{sign}$ acting by pointwise multiplication then

$$\{\not{D}, \gamma\} = 0.$$

This should not be viewed as chirality since it acts on the spinor components as functions not on the spinor values. It does, however, mean that solutions are paired with massive eigenvalue m going to eigenvalue $-m$ under γ. We define mass here as the negative eigenvalue of $\not{D}$. Note, however, that $\not{D}^2$ is a second-order operator (it involves R_{uv}, R_{vu}) and not merely $\square$ plus a scalar curvature term as in the Lichnerowicz formula.

Proposition 2.2. *$\not{D}$ has four zero modes, four massive modes with eigenvalue* $+1$ *and four with eigenvalue* -1, *related by* γ.

Proof. To find the solutions we consider first of all spinors of the form

$$\psi = \begin{pmatrix} R_{uv}\phi \\ \phi \end{pmatrix}$$

for some function $\phi \in \mathbb{C}[S_3]$. The Dirac operator reduces to

$$\not{D}\psi = -\frac{1}{3}\sum_a R_a\psi = \left(-1 - \frac{1}{6}\square\right)\psi$$

acting on each component. Hence there are four linearly independent zero modes of the form

$$\psi_{ij} = \begin{pmatrix} R_{uv}\phi_{ij} \\ \phi_{ij} \end{pmatrix}$$

induced by the spin 0 waves ϕ_{ij} of mass $\sqrt{6}$. We also have a massive mode of eigenvalue -1 from $\phi = 1$ and $+1$ from $\phi =$ sign from the remaining spin 0 waves, but these solutions are obvious by inspection. In fact it is obvious that

$$\psi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \psi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are separately solutions of eigenvalue -1, and similarly when multiplied by sign for eigenvalue $+1$.

Two further and independent solutions of eigenvalue -1 are obtained by the similar ansatz

$$\psi = \begin{pmatrix} \phi \\ R_{uv}\phi \end{pmatrix}.$$

This time

$$\not{D}\psi = \begin{pmatrix} \Delta\phi \\ R_{uv}\Delta\phi \end{pmatrix}, \qquad \Delta = R_v - \frac{2}{3}\sum_a R_a$$

which is easily solved by ϕ a linear combination of the ϕ_{ij}. The second term of Δ vanishes on these and $R_v\phi_{ij} = \phi_{ik}\rho(v)^k{}_j$. But $\rho(v)$ has precisely one eigenvector α of eigenvalue -1

and hence contracting with this gives a pair of solutions $\phi = \phi_{ij}\alpha^j$ of eigenvalue -1. In the basis used above, the resulting two massive spinor waves of eigenvalue -1 are

$$\psi_i = \begin{pmatrix} \phi_{i2} \\ R_{uv}\phi_{i2} \end{pmatrix}.$$

They are linear independent since ρ was irreducible. Similarly, for eigenvalue $+1$ if we use the $+1$ eigenvector of $\rho(v)$. Altogether we have a complete diagonalisation of $\not{D}$. □

We can consider real or complex spinors (in fact the linear theory works over any field of characteristic 0). For a general group G any irreducible representation ρ similarly defines γ_a-matrices [3] and one can expect a similar method to the above to diagonalise $\not{D}$ with mass spectrum related to the eigenvalues of $\mathcal{C}$ in the representation.

2.3. Zero curvature Maxwell fields and deRham cohomology

For a spin 1 or Maxwell 'photon' field we take a 1-form $A \in \Omega^1$ defined modulo exact differentials or 'linearised gauge transformations'. The well-defined curvature is of course

$$F = \mathrm{d}A. \tag{2}$$

For example, the moduli space of flat connections modulo gauge transformations in this linearised context is the cohomology H^1 with respect to the noncommutative differential forms.

Proposition 2.3. *The noncommutative deRham cohomology of S_3 is*

$$H^0 = \mathbb{C}.1, \qquad H^1 = \mathbb{C}.\theta, \qquad H^2 = 0, \qquad H^3 = \mathbb{C}.\star\theta, \qquad H^4 = \mathbb{C}.\mathrm{Top}$$

and exhibits Poincaré duality.

Proof. Here a closed 0-form means f with $\partial^a f = 0$ for all a, which means $R_a(f) = f$ for all a. But $a \in \mathcal{C}$ generate all of S_3 so it means a multiple of 1. For H^1 we consider a 1-form $A = A^a e_a$ with components A^a. Each has six values. Similarly, we take our basis for Ω^2 with

$$F^{uv} = R_u A^v + A^u - R_w A^u - A^w, \qquad F^{vu} = R_v A^u + A^v - R_u A^w - A^u,$$
$$F^{vw} = R_v A^w + A^v - R_w A^u - A^w, \qquad F^{wv} = R_w A^v + A^w - R_u A^w - A^u$$

for the components in our basis. Hence d is a 24×18 matrix

$$\mathrm{d}_1 = \begin{pmatrix} \mathrm{id} - R_w & R_u & -\mathrm{id} \\ R_v - \mathrm{id} & \mathrm{id} & -R_u \\ -R_w & \mathrm{id} & R_v - \mathrm{id} \\ -\mathrm{id} & R_w & \mathrm{id} - R_u \end{pmatrix}.$$

We find its kernel, which contains in particular the five independent exact differentials $\mathrm{d}\delta_x$ ($x \neq e$, say) to be 6D. Hence $H^1 = \mathbb{C}$. It is easy to see that it is represented by θ which is closed but not exact. Next, the image of d above must be 12D. For $\mathrm{d}:\Omega^2\to\Omega^3$ we similarly compute

$$\begin{aligned}\mathrm{d}F&=(\partial^w(F^{uv}-F^{vw})-\partial^vF^{wv}+F^{uv}-F^{vu})\tfrac{1}{2}\star e_u\\&\quad+(\partial^wF^{vu}+\partial^uF^{vw}+F^{vu}-F^{uv}+F^{vw}-F^{wv})\tfrac{1}{2}\star e_v\\&\quad+(\partial^u(F^{wv}-F^{vu})-\partial^vF^{uv}+F^{wv}-F^{vw})\tfrac{1}{2}\star e_w.\end{aligned}$$

We use here

$$\mathrm{d}(e_a\wedge e_b)=\tfrac{1}{2}(\star(e_a)-\star(e_b))$$

and the relations in Ω^3. The result can be written as

$$\begin{aligned}\mathrm{d}F&=(R_w(F^{uv}-F^{vw})-R_vF^{wv}+F^{vw}+F^{wv}-F^{vu})\tfrac{1}{2}\star e_u\\&\quad+(R_wF^{vu}+R_uF^{vw}-F^{uv}-F^{wv})\tfrac{1}{2}\star e_v\\&\quad+(R_u(F^{wv}-F^{vu})-R_vF^{uv}+F^{uv}+F^{vu}-F^{vw})\tfrac{1}{2}\star e_w\end{aligned}$$

which is the 18×24 matrix

$$\mathrm{d}_2=\begin{pmatrix}R_w & -\mathrm{id} & \mathrm{id}-R_w & \mathrm{id}-R_v\\ -\mathrm{id} & R_w & R_u & -\mathrm{id}\\ \mathrm{id}-R_v & \mathrm{id}-R_u & -\mathrm{id} & R_u\end{pmatrix}$$

which is basically the transpose of the matrix above for d_1. Hence its kernel is 12D and $H^2=0$. It also means that the dimension of the space of exact 3-forms as 12. Next, for H^4 we look at d on our 3-forms. Thus,

$$\mathrm{d}\star e_u=2\,\mathrm{d}(e_w\wedge e_u\wedge e_v)=2e_u\wedge e_w\wedge e_u\wedge e_v+2e_w\wedge e_u\wedge e_v\wedge e_u=0$$

hence $\mathrm{d}f^a\star(e_a)=\partial^be_b\wedge\star(e_a)=2(\partial^af^a)\mathrm{Top}$ is the image of d for any three functions f^a. The 6×18 matrix of d on (f^u,f^v,f^w) is evidently the transpose of the matrix for d on functions, hence its image is 5D. Note that this image is precisely the space of functions with zero integral over S_3 (times Top). Thus, $H^4=\mathbb{C}$ and is represented by a constant multiple of the Top form. Moreover, the kernel of $\mathrm{d}:\Omega^3\to\Omega^4$ is therefore, 13D, hence $H^3=\mathbb{C}$. It is easy to see that it is represented by $\star\theta$. In particular, we find Poincaré duality as stated. □

One may similarly prove the Hodge decomposition of forms in each degree into a direct sum of exact, coexact and harmonic forms, where harmonic means closed and coclosed as defined by $\star$.

2.4. Spin 1: Maxwell equations

We now look at the wave operator for spin 1 or 'Maxwell fields' A modulo exact forms. Here the invariant curvature $F=\mathrm{d}A$ is a linear version of the true $U(1)$ gauge theory in the next section. In noncommutative geometry, the latter looks and behaves more like Yang–Mills theory while the linear theory is more like conventional electromagnetism.

We note that

$$(\mathrm{d}A)^* = ((R_a A^b + A^a)e_a \wedge e_b)^* = e_b \wedge e_a(R_a \bar{A}^b + \bar{A}^a)$$
$$= (R_b \bar{A}^b + R_{ba} A^a)e_b \wedge e_a = (A^{*b} + R_b A^{*a})e_b \wedge e_a = \mathrm{d}(A^*)$$

as it should. Note that in our basis we have

$$A^{*a} = R_a(\bar{A}^a), \qquad F^{*ab} = R_{ab}\bar{F}^{ba}.$$

Then up to total derivatives

$$L\,\mathrm{Top} \equiv -\frac{\sqrt{3}}{4}F^* \wedge \star F = -\frac{\sqrt{3}}{4}(\mathrm{d}A)^* \wedge \star(\mathrm{d}A) = -\frac{\sqrt{3}}{4}A^* \wedge \mathrm{d}\star\mathrm{d}A$$

gives the Lagrangian and the required wave operator

$$\star\mathrm{d}\star\mathrm{d} : \Omega^1 \to \Omega^1.$$

Note that $\mathrm{d}(f^a\star(e_a)) = 2(\partial^a f^a)\mathrm{Top}$ and $\int \partial^a f^a = 0$ means that we can indeed neglect exact 4-forms in these computations, as we do.

One may also write the Maxwell action more explicitly. Thus

$$^\star F^{uv} = -F^{uv} + 2F^{vw}, \qquad ^\star F^{vu} = F^{vu} - 2F^{wv},$$
$$^\star F^{vw} = F^{vw} - 2F^{uv}, \qquad ^\star F^{wv} = -F^{wv} + 2F^{vu}$$

from which

$$L = -\tfrac{1}{4}(\bar{F}^{uv}(F^{vw} - 2F^{uv}) + \bar{F}^{vu}(F^{wv} - 2F^{vu})$$
$$+ \bar{F}^{vw}(F^{uv} - 2F^{vw}) + \bar{F}^{wv}(F^{vu} - 2F^{wv})) \tag{3}$$

using the relations in Ω^4 and up to total derivatives. This is

$$L = \tfrac{1}{2}(|F^{uv}|^2 + |F^{vu}|^2 + |F^{vw}|^2 + |F^{wv}|^2 - \mathrm{Re}(\bar{F}^{uv}F^{vw} + \bar{F}^{vu}F^{wv}))$$

from which the action is easily seen to be positive semidefinite. Also, it is tempting to divide F into two halves related through $\star$ much as in the theory of electromagnetism. One such division is

$$E = (F^{uv}, F^{vu}), \qquad B = (F^{vw}, F^{wv})$$

since E and B are then rotated componentwise into each other by $\star$. The action is then the sum of similar parts from E and from B and a cross term.

Proposition 2.4. *The zero modes of the wave operator* $\star\mathrm{d}\star\mathrm{d}$ *are precisely the fields of zero curvature. The equations*

$$\mathrm{d}F = 0, \qquad \star\mathrm{d}\star F = J$$

have a solution iff J is 'strongly conserved' in the sense $\mathrm{d}\star J = 0$ *and* $\int J \wedge \star\theta = 0$, *and the solution F is unique. The space of possible sources is 12D and spanned by four massive* $\star\mathrm{d}\star\mathrm{d}$ *modes for each of the masses* $\sqrt{3}$, $\sqrt{6}$ *and* 3.

Proof. Putting in the form of $F = \mathrm{d}A$ into the general formulae for $\star F$ and d on Ω^2 (as given in the cohomology computation) we obtain $\star\mathrm{d}\star\mathrm{d}A$ with e_u component

$$R_{uv}A^v + R_{vu}A^w - R_u(A^v + A^w) + R_v(A^u - A^v + A^w) + R_w(A^u + A^v - A^w) - 4A^u + A^v + A^w = 0$$

and its two cyclic rotations. Equivalently, the matrix for $\star$ on 2-forms in our standard basis is

$$\star_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} -1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 \\ -2 & 0 & 1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix}$$

and as a matrix on the column vector of the components of A,

$$\star\mathrm{d}\star\mathrm{d} = \mathrm{d}_2\star_2\mathrm{d}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} R_v+R_w-4 & R_{uv}-R_u-R_v+R_w+1 & R_{vu}-R_u+R_v-R_w+1 \\ R_{vu}-R_u-R_v+R_w+1 & R_u+R_w-4 & R_{uv}+R_u-R_v-R_w+1 \\ R_{uv}-R_w-R_u+R_v+1 & R_{vu}-R_w+R_u-R_v+1 & R_v+R_u-4 \end{pmatrix}.$$

This 18×18 matrix has a 6D kernel which is the kernel of $\mathrm{d} : \Omega^1 \to \Omega^2$ as in Proposition 2.3, i.e. it is precisely the closed forms or forms of zero curvature. It means that if we solve $\star\mathrm{d}\star\mathrm{d}A = J$ for F rather than for A we have exactly one solution for each J in the image of the wave operator. The image is therefore, 12D which is the dimension of the image of $\mathrm{d} : \Omega^2 \to \Omega^3$ in the cohomology computation, i.e. we require precisely that $\star(J)$ be exact. On the other hand, for any 2-form F, $\star\mathrm{d}F$ as given in the proof of Proposition 2.3 is such that $\star\mathrm{d}F \wedge \star\theta$ is an exact 4-form. Indeed, its components are given by adding up the coefficients of $\star e_a$ in $\star\mathrm{d}F$, which add up to a total derivative. This additional property characterises exact 3-forms in the 13D space of closed 3-forms. Hence in our case $\star J$ exact is therefore characterised by $\mathrm{d}\star J = 0$ and $J \wedge \star\theta$ an exact 4-form. The latter is the condition that its integral as a 4-form (which means the usual integral of the coefficient of Top) be 0.

Finally, the other eigenvalues of $\star\mathrm{d}\star\mathrm{d}$ are easily found using the above matrix representation to be -3, -6 and -9 corresponding to a massive mode as stated. The application of $\star\mathrm{d}\star\mathrm{d}$ to these gives the space of possible sources. Each eigenspace is 4D and together with the zero modes they fully diagonalise $\star\mathrm{d}\star\mathrm{d}$. □

The two conditions for a strongly conserved source can be written explicitly as

$$\sum_a \partial^a J^a = 0, \qquad \int \sum_a J^a = 0 \tag{4}$$

and the second is equivalent to $\sum_a J^a$ a total derivative. This is stronger than just the usual zero divergence condition alone precisely due to a nontrivial H^3. Other than this complication (which can arise in the continuum case just as well) we see that there is a reasonable theory of 'electromagnetism' or 'electrostatics'. The explicit form of the equations for F are the Bianchi equation $\mathrm{d}F = 0$ given explicitly in the proof of Proposition 2.3 and $\star\mathrm{d}\star F = J$, which after adding or subtracting the respective Bianchi identities comes out as

$$\begin{aligned} J^u &= \partial^w F^{uv} - \partial^v F^{vu} - F^{vu} + F^{vw} + F^{wv}, \\ J^v &= -\partial^u F^{uv} - \partial^w F^{wv} - F^{vu} - F^{vw}, \\ J^w &= \partial^u F^{wv} - \partial^v F^{vw} + F^{uv} + F^{vu} - F^{vw}. \end{aligned}$$

And if one want the potential A, this is determined only up to zero modes. These can be gauge fixed by similarly restricting A to *strong Coulomb gauge*

$$\sum_a \partial^a A^a = 0, \qquad \int \sum_a A^a = 0. \tag{5}$$

It remains to construct suitable currents J of a recognisable form from such a point of view. We obtain them by considering scalar fields of mass m.

Proposition 2.5. *If ϕ is an on-shell scalar field of mass m then*

$$J^a = (\partial^a\bar{\phi})\phi - (R_a\bar{\phi})\partial^a\phi + \frac{m^2}{18}\int\bar{\phi}\phi = 2\partial^a(\bar{\phi})\phi - \partial^a(\bar{\phi}\phi) + \frac{m^2}{18}\int\bar{\phi}\phi$$

is a strongly conserved current.

Proof. Here the 'local' term is obtained by minimal coupling, i.e. from expanding $((\mathrm{d} + A)\phi)^* \wedge (\mathrm{d} + A)\phi$ and has zero divergence. The m^2 term does not change this fact but ensures conservation in our strong sense. Thus, from the braided-Leibniz rule we have

$$\sum_a \partial^a J^a = \sum_a (\partial^a\partial^a\bar{\phi})\phi - \sum_a \bar{\phi}\partial^a\partial^a\phi = -(\Box\bar{\phi})\phi + \bar{\phi}(\Box\phi) = 0$$

when ϕ is on shell (an eigenvector of the wave operator). And

$$\sum J^a = (\Box\bar{\phi})\phi - \frac{1}{2}\Box(\bar{\phi}\phi) + \frac{m^2}{6}\int\bar{\phi}\phi$$

which has integral zero. The middle term is a total derivative and does not contribute. $\Box$

Hence we have a strongly conserved current for any on-shell solution ϕ of the wave equation. The mass $0, 2\sqrt{3}$ solutions from Section 2.1 have zero current. The mass $\sqrt{6}$ modes, however, have a nonzero current. We use the projection given there of these modes from functions ϕ_0 and take for these the 'point source' form δ_x. Then the corresponding 'point-like' mass $\sqrt{6}$ modes are

$$\phi = 2\delta_x - \delta_{xuv} - \delta_{xvu}.$$

Here

$$\bar{\phi}\phi = 4\delta_x + \delta_{xuv} + \delta_{xvu}, \qquad R_a(\bar{\phi})\phi = 0$$

so that we obtain the current for a ‘point-like source’ at x

$$J_x^a = 2 - R_a(\bar{\phi}\phi) - \bar{\phi}\phi = 1 - 3\delta_x - 3\delta_{xa}. \tag{6}$$

These sources are ‘radial’ in the sense that the component J^a in the a direction of the source located at x has support along the line x, xa (plus an overall constant value).

These point-like sources at the different x are not independent. It is easy to see that $J_{xu} + J_{xv} + J_{xw} = 0$ so three point-like sources symmetrically placed about any point cancel out. Indeed, the above construction gives only four independent sources due to the two relations

$$J_u + J_v + J_w = 0, \quad J_e + J_{uv} + J_{vu} = 0. \tag{7}$$

In fact, these point-like sources span the 4D, -6 eigenspace of $\star\mathrm{d}\star\mathrm{d}$ which means that the corresponding potential for a source at x in ‘strong Coulomb gauge’ is simply

$$A_x = -\tfrac{1}{6}J_x.$$

Its curvature F may then be easily computed as

$$\begin{aligned} F^{uv} &= \delta_{xu} - \delta_{xw}, \qquad & F^{vu} &= \delta_{xv} - \delta_{xu}, \\ F^{vw} &= \delta_{xv} - \delta_{xw}, \qquad & F^{wv} &= \delta_{xw} - \delta_{xu}. \end{aligned} \tag{8}$$

Next we consider ‘dipole’ configurations. We can clearly polarise the above formula for J for a scalar field as $J(\phi, \psi)$ where one ϕ is replaced by an independent field ψ say. We still have a strongly conserved source as long as ϕ, ψ are on shell with the same mass. Here

$$J(\phi, \psi) + J(\psi, \phi) = J(\phi + \psi) - J(\phi) - J(\psi)$$

is the source for the combined field minus the source for each field separately. Letting ϕ, ψ be two ‘point-like’ solutions at x, xb, respectively (with $b \in \mathcal{C}$), i.e. a ‘dipole’ at x in direction b, we have $\bar{\phi}\psi = 0$ and

$$J^a_{x;b} = 2R_a(\bar{\phi})\psi = (9\delta_{a,b} - 6)(\delta_{xa} + \delta_{xb}) + 2\sum_{c\in\mathcal{C}}\delta_{xc}.$$

Here the current is positive when ‘lined up’ with b. This is our first attempt at a dipole source. Note that there are only four independent sources due to the relations:

$$J^a_{x;b} = R^a J^a_{xb;b}, \qquad J^a_{x;u} + J^a_{x;v} + J^a_{x;w} = 0, \qquad J^a_{x;b} + J^a_{x(uv);b(uv)} + J^a_{x(uv)^2;b(uv)^2} = 0 \tag{9}$$

and one may find the corresponding potential as

$$A^a_{x;b} = -\tfrac{1}{9}(2J^a_{x;b} + R^a J^a_{x;b}).$$

Starting from this source, one can then find nicer formulae if one introduces a slightly modified source (still satisfying the strong conservation conditions)

$$J'^{a}_{x;b} = J^{a}_{x;b} + \tfrac{1}{2}\partial^{a} J^{a}_{x;b} = \tfrac{1}{2}(J^{a}_{x;b} + J^{a}_{xb;b})$$

using the first of the relations (9). Explicitly

$$J'^{a}_{x;b} = 1 + \tfrac{1}{2}(9\delta_{a,b} - 6)(\delta_x + \delta_{xa} + \delta_{xb} + \delta_{xab}). \tag{10}$$

As before there are four independent configurations here and they span the eigenspace of $\star\mathrm{d}\star\mathrm{d}$, now of eigenvalue -3. The corresponding dipole potential is therefore

$$A^{a}_{x;b} = -\tfrac{1}{3} J'^{a}_{x;b}.$$

Its curvature can easily be computed and one finds for a dipole centred at x and directed along $b = u$ (say)

$$F^{uv}_{x;u} = 9(\delta_{xu} - \delta_{xvu} + \delta_x - \delta_{xw}), \qquad F^{vu}_{x;u} = 9(\delta_{xuv} - \delta_{xu} + \delta_{xv} - \delta_x) \tag{11}$$

$$\begin{aligned} F^{vw}_{x;u} &= 9(-\delta_{xvu} - \delta_{xv} - \delta_{xuv} - \delta_{xw} - \delta_{xuv} - \delta_{xv}),\\ F^{wv}_{x;u} &= 9(\delta_{xu} - \delta_{xvu} + \delta_{xw} - \delta_x - \delta_{xw} - \delta_{xvu}). \end{aligned} \tag{12}$$

This gives an electrostatics picture of some of the massive spin one modes. Note that the mass here, as for the lower spins, reflects the background constant curvature of S_3 in the sense of [3].

3. $U(1)$ noncommutative Yang–Mills theory

Here we do $U(1)$ 'gauge theory' in the more usual sense. In usual commutative geometry this essentially coincides with cohomology theory but in the noncommutative case the curvature

$$F = \mathrm{d}A + A \wedge A \tag{13}$$

remains nonlinear. It is covariant as $F \mapsto UFU^{-1}$ under

$$A \mapsto UAU^{-1} + U\,\mathrm{d}U^{-1}, \qquad A^{a} \mapsto \frac{U}{R_a(U)} A^{a} + U\partial^{a} U^{-1} \tag{14}$$

for any unitary U (i.e. any function of modulus 1). Here we limit attention to 'real' A in the sense $A^{*} = A$. This translates in terms of components as

$$\bar{A}^{a} = R_a A^{a}, \qquad \bar{F}^{ab} = R_{ab}(F^{ba})$$

and implies that $F^{*} = F$ is 'real'.

Our first step is to change variables to $A = \Phi - \theta$, i.e. $A^{a} = \Phi^{a} - 1$ and certain operators $\rho_a \equiv \Phi^{a} R_a$

$$F^{uv} = \rho_u \Phi^{v} - \rho_w \Phi^{u}, \qquad F^{vu} = \rho_v \Phi^{u} - \rho_u \Phi^{w}, \tag{15}$$

$$F^{vw} = \rho_v \Phi^{w} - \rho_w \Phi^{u}, \qquad F^{wv} = \rho_w \Phi^{v} - \rho_u \Phi^{w}. \tag{16}$$

Here $\Phi^a \mapsto (U/R_a U)\Phi^a$ transforms covariantly and

$$\bar{\Phi}^a = R_a \Phi^a \tag{17}$$

is our reality constraint. The reality constraint means that Φ^a are determined freely by their values on u, v, w. It also means that

$$\lambda_a^2 \equiv |\Phi^a|^2 \tag{18}$$

are real-valued gauge-invariant functions associated to any gauge field.

3.1. Zero curvature moduli space

In classical geometry the zero curvature gauge fields detect the 'homotopy' or fundamental group of a manifold. Hence in noncommutative geometry the presence of a moduli of flat connections is indicative of this. We find it to be nontrivial.

Theorem 3.1. *The moduli space of zero curvature gauge fields modulo gauge transformation is the union of a 1-parameter positive half-line*

$$A = (\mu - 1)\theta, \quad \mu \geq 0$$

and six positive cones of $\mathbb{R}^3$ *of the form*

$$A = \Phi - \theta, \quad \Phi^a(b) = \mu^a{}_b, \quad a, b \in \mathcal{C},$$

where $\mu^a{}_b \geq 0$ *are a matrix of the form*

$$\text{(i)}: \quad \begin{pmatrix} * & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ * & * & * \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ * & * & * \end{pmatrix},$$

$$\text{(ii)}: \quad \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}.$$

Proof. Given any zero curvature solution we clearly have

$$\rho_u \Phi^v = \rho_w \Phi^u = \rho_v \Phi^w, \qquad \rho_u \Phi^w = \rho_v \Phi^u = \rho_w \Phi^v.$$

In fact these two equations are equivalent under the reality assumption but it is useful to work with both forms. Then

$$\begin{aligned} \rho_u \rho_u \Phi^v &= \rho_u(\Phi^u R_u \Phi^v) = \Phi^v \lambda_u^2 = \rho_u \rho_w \Phi^u = \rho_u(\Phi^w) R_{vu}(\Phi^u) = \Phi^v R_v(\lambda_u^2) \\ &= \rho_u \rho_v \Phi^w = \rho_u(\Phi^v) R_{uv} \Phi^w = \rho_v(\Phi^w) R_{vw}(\Phi^w) = \Phi^v R_v(\lambda_w^2). \end{aligned}$$

Hence

$$\lambda_v^2(\lambda_w^2 - \lambda_u^2) = 0, \qquad \lambda_v^2 \partial^v \lambda_u^2 = 0, \qquad \lambda_v^2 \partial^v (\lambda_w^2) = 0$$

and the cyclic rotations of these. We also have $\partial^v\lambda_v^2 = 0$, etc. Choose any point (for a nonzero configuration) where $\Phi^v(x) \neq 0$, say. Then $\lambda_u(x) = \lambda_w(x)$ so either both are zero or not. Assume the latter (so all components at x are nonzero). Then $\lambda_u(xv) = \lambda_w(xv) \neq 0$ since $\lambda_v(x) \neq 0$, and $\lambda_v(xv) = \lambda_w(xv)$ since $\lambda_u(xv) \neq 0$, hence all components at xv are also nonzero. Iterating, we conclude in this case that $\lambda_u^2 = \lambda_v^2 = \lambda_w^2 = \mu^2$ say, where μ is a positive constant. The other possibility is that at every $x \in S_3$ at most one component is nonzero, which degenerate case will be handled later.

In the nowhere zero case, we consider the gauge transform

$$U(e) = 1, \qquad U(u) = \frac{\Phi^u(e)}{\mu}, \qquad U(v) = \frac{\Phi^v(e)}{\mu}, \qquad U(w) = \frac{\Phi^w(e)}{\mu},$$
$$U(uv) = \frac{\Phi^u(e)\Phi^v(u)}{\mu^2}, \qquad U(vu) = \frac{\Phi^w(e)\Phi^v(w)}{\mu^2}$$

which is manifestly unitary. Using the zero curvature conditions one may check that indeed it gauge transforms Φ to $\Phi^u = \Phi^v = \Phi^w = \mu$.

We turn now to the degenerate case where at each point at most one component of Φ is nonzero. Note first that we need only be concerned with the matrix $\{\Phi^a(b)\}$, where a, b run over u, v, w, since the reality condition determines the values then at e, uv, vu. Moreover, the reality condition then becomes empty. For example, $\Phi^u(uv) = \Phi^u(wu) = \bar{\phi}^u(w)$ and $\Phi^v(uv) = \bar{\Phi}^v(u)$, etc. Next, under a gauge transform this matrix goes to

$$\Phi^a{}_b \to \Phi^a(b)\frac{U(b)}{U(ba)}$$

and because under our degeneracy assumption at most one entry in each column is nonzero, we can choose this in such a way that all nonzero entries can be gauge transformed onto the real positive axis. Indeed, we chose

$$U(e) = U(uv) = U(vu) = 1, \qquad U(b) = \frac{|\Phi^a(b)|}{\Phi^a(b)},$$

where there is at most one nonzero $\Phi^a(b)$ at each $b = u, v, w$ (and we set $U(b) = 1$ if there is none). Thus every zero curvature solution of our degenerate type is gauge equivalent to one where the matrix is given by real nonnegative numbers $\mu^a{}_b$ of at most three entries. These are equal to the gauge-invariant norms λ_a^2 and cannot be transformed further while remaining on the positive real axis, so there is one solution for each allowed matrix.

Precisely which matrices are allowed is determined by the zero-curvature equation. Writing this out in terms of the $\Phi^a(b)$ we have

$$\Phi^u(u)\Phi^v(v) = \Phi^v(u)\Phi^w(v) = \Phi^w(u)\Phi^u(v),$$
$$\Phi^u(v)\Phi^v(w) = \Phi^v(v)\Phi^w(w) = \phi^w(v)\Phi^u(w),$$
$$\Phi^u(w)\Phi^v(u) = \Phi^v(w)\Phi^w(u) = \Phi^w(w)\Phi^u(u)$$

for the zero curvature at u, v, w. At the other points it yields

$$\Phi^v(u)\Phi^u(u) = \Phi^w(v)\Phi^v(v) = \Phi^u(w)\Phi^w(w),$$
$$\Phi^v(w)\Phi^u(w) = \Phi^w(u)\Phi^v(u) = \phi^u(v)\Phi^w(v),$$
$$\Phi^v(v)\Phi^u(v) = \Phi^w(w)\Phi^v(w) = \Phi^u(u)\Phi^w(u)$$

which is empty in our case where every column has at most one nonzero entry (it is the origin of this restriction).

Finally, we enumerate the allowed patterns. (i) Clearly if two rows (i.e. two of the Φ^u, Φ^v, Φ^w are entirely zero) then the third is free for a zero curvature solution. This is the first set of matrices shown. (ii) If exactly one row is entirely zero, say Φ^w, then the other two obey

$$\Phi^u(u)\Phi^v(v) = 0, \qquad \Phi^u(v)\Phi^v(w) = 0, \qquad \Phi^u(w)\Phi^v(u) = 0$$

from the first zero of zero-curvature equations. This says that the Φ^u row has no nonzero entries with the rotated Φ^v row. If one row has more than one nonzero entry then this forces the other row to be entirely 0 as well and we are back in case (i). Otherwise, neither row can have more than one nonzero entry which means that we are either in case (i) again or in a degenerate case of the next case. (iii) The remaining case is when each row Φ^a has at most one nonzero entry $\Phi^a(\sigma(a))$, say, for some permutation σ of u, v, w (anything else would imply one of the rows was entirely zero, covered above). In this case we have potentially six possibilities depending on $\sigma \in S_3$. Now, for this type of solution the zero-curvature equation reads

$$\Phi^a(\sigma(a))\Phi^b(\sigma(b)) = 0, \quad \text{if } \sigma(a)\sigma(b) = ab.$$

For $\sigma = \mathrm{id}$ this means $\Phi^a(a)\Phi^b(b) = 0$ for all a, b, which means that two out of three of our rows must be zero, which puts us back in case (i) above. Similarly, if σ is a rotation $u \to v \to w \to u$ or its inverse then we have three equations forcing two out of three to be 0 and hence in case (i). The three remaining possibilities are where σ fixes one of u, v, w and flips the other two. In this case the relations are empty, i.e. we can freely chose the potentially nonzero matrix entries $\Phi^a(\sigma(a))$. This is the second family of positive cones in $\mathbb{R}^3$ stated. Note that the matrices of u, v, w themselves in their natural representation on three elements are in this second family. □

Similarly, in terms of the components of F and $\star F$ as in the previous section, we have the self-duality equation as

$$F^{vw} = \lambda F^{uv}, \qquad F^{wv} = \lambda^{-1} F^{vu}, \qquad \lambda = \tfrac{1}{2}(1 + \mathrm{i}\sqrt{3})$$

after collecting terms. Note that $|\lambda| = 1$ and $\lambda^3 = -1$. Under our reality condition only one of these equations is needed, the other being equivalent. We see that a self-dual 2-form subject to our reality condition is therefore determined entirely by an unconstrained complex function F^{uv}.

One could therefore ask for the moduli of self-dual gauge fields or 'instantons', i.e. when such 2-forms can be the curvature of a gauge field. Note that there can be no self-dual

Maxwell connections other than $F = 0$ due to the unique solution of the Maxwell equations for F with no source (as seen in the preceding section). Therefore, one should not necessarily expect instantons here either. Indeed, putting in the form of F for the $U(1)$ Yang–Mills theory, we obtain the self-duality equations as

$$\rho_u \Phi^v = \lambda^{-1}\rho_v \Phi^w + \lambda \rho_w \Phi^u$$

and our 'reality' constraint on the Φ. This appears to have no solutions.

3.2. Yang–Mills action and other extrema

Finally, we take a look at the Yang–Mills action in general. In terms of F the Lagrangian is exactly the same as that stated in Section 2.4 for the Maxwell field, and is therefore positive semidefinite. In our Yang–Mills case we put in the form of F in terms of Φ.

Theorem 3.2. *The (rescaled) Yang–Mills action in terms of the gauge field fluctuation Φ and up to total derivatives is*

$$L = -\frac{\sqrt{3}}{4} F^* \wedge \star F = \lambda_u^2 R_u \lambda_v^2 - \Phi^u (R_u \Phi^v)(R_{uv}\Phi^u)(R_w \Phi^w) + \text{cyclic}$$

and is positive semidefinite.

Proof. We put the form of F into the second expression for the Lagrangian in Section 2.4. First, we explicitly put in the reality condition on the F which implies that

$$L = |F^{uv}|^2 + |F^{vw}|^2 - \mathrm{Re}(\bar{F}^{uv} F^{vw})$$

up to a total derivative. Then

$$\begin{aligned} |F^{uv}|^2 &= R_{uv}(\Phi^v R_v \Phi^u - \Phi^u R_u \Phi^w)(\Phi^u R_u \Phi^v - \Phi^w R_w \Phi^u) \\ &= \lambda_u^2 R_u \lambda_v^2 + \lambda_w^2 R_w \lambda_u^2 - 2\Phi^w (R_w \Phi^u)(R_u \Phi^u) R_{uv} \Phi^v \end{aligned}$$

up to a total derivative. Similarly

$$|F^{vw}|^2 = \lambda_v^2 R_v \lambda_w^2 + \lambda_w^2 R_w \lambda_u^2 - 2\Phi^w (R_v \Phi^v)(R_w \Phi^u) R_{uv} \Phi^w.$$

Finally, we compute

$$\begin{aligned} \bar{F}^{uv} F^{vw} = {} & (R_u \Phi^u)\Phi^v (R_v \Phi^w) R_{uv} \Phi^v + \lambda_w^2 R_w \lambda_u^2 \\ & - (R_w \Phi^w)\Phi^v (R_v \Phi^w) R_{uv} \Phi^u - (R_u \Phi^u)\Phi^w (R_w \Phi^u) R_{uv} \Phi^v. \end{aligned}$$

Adding minus the real part of this to the other terms and discarding total derivatives gives the result for L. □

From the physical point of view this result is very significant. It states that when we write the values of $\Phi^a(x)$ in polar coordinates their gauge-invariant fields $\lambda_a^2(x)$ contribute like some kind of massive particle with Lagrangian

$$L_0 = \lambda_u^2 \partial^u \lambda_v^2 + \lambda_v^2 \partial^v \lambda_w^2 + \lambda_w^2 \partial^w \lambda_u^2 + \lambda_u^2 \lambda_v^2 + \lambda_v^2 \lambda_w^2 + \lambda_w^2 \lambda_u^2 \tag{19}$$

and a part given by the sum of the Wilson loops W_u, W_v, W_w at x. Here

$$W_u = \Phi^u(R_u\Phi^v)(R_{uv}\Phi^u)(R_w\Phi^w),$$
$$W_u(x) = \Phi^u(x)\Phi^v(xu)\Phi^u(xuv)\Phi^w(xw) \tag{20}$$

is the product around a path defined by right translating by u, then by v then by u ($= u^{-1}$) and then by w. Here $uvuw = e$ is a relation in S_3 in terms of our elements of $\mathbb{C}$ and such relations form our elementary plaquettes. One can also introduce homology and homotopy of allowed paths in the group as defined by filling in via elementary plaquettes, i.e. one should think of them as 'pieces of area' defined by the differential calculus.

We will say more about the $U(1)$ lattice gauge theory defined by the angular part of $\Phi = \lambda\,\mathrm{e}^{\mathrm{i}\theta}$ in the next section. At present we concentrate on the real-positive radial variables λ with 'free particle' Lagrangian $L_0(\lambda)$ (which is quadratic in terms of the functions λ_a^2). Note that $\lambda_a(x) = \lambda_a(xa)$, i.e. these variables are really associated to the steps (edges) along allowed directions in the lattice. They are a hybrid of some kind of 'length' or 'metric' assignment to the abstract lattice on which the more conventional $U(1)$ gauge theory takes place, and the real part of the field strength of A (they are the modulus of the infinitesimal transport '$1 + A^a(x)\,\mathrm{d}x_a$' and hence involve both features rolled into one). The noncommutative Yang–Mills theory factorises into some kind of 'metric' theory for the λ and a conventional lattice $U(1)$ for the angular variables. Apart from L_0 there is also an interaction term coming from the polar decomposition

$$W_u(x) = \lambda_u(x)\lambda_v(xu)\lambda_u(xuv)\lambda_w(xw)w_u(x),$$

where w_u, etc. are the conventional $U(1)$-valued Wilson loops. One may heuristically think of expressions such as $\lambda_u^2\lambda_v^2$ in L_0 as 'area' of an elementary plaquette and the products of the λ in the W_u as 'multiplicative perimiter'. It is interesting that both expressions are quartic, which is consistent with the idea that holonomies in finite lattice theory go as area law (this would becomes Wilson's criterion for confinement if it survived to the continuum limit, but we are not able to consider this in our finite model). Note also that a flat connection A corresponds to both a flat $U(1)$ connection in the sense of trivial holonomy around the elementary plaquettes as above and a flat assignment of the λ variables when multiplied. The physical meaning of this is not clear (it comes from the field strength nature of the λ and perhaps suggests to think of them as transition probabilities when suitably normalised). At any rate, one has a real $\mathbb{R}_+$-valued gauge theory for the λ in the finite geometry as well as a $U(1)$ lattice theory. These are quite general features that apply for other groups also.

In particular, we can look at the pure 'metric' sector of the theory where all the $U(1)$-Wilson loops w_a are constrained to be 1. For example, we can take all the Φ real. In any case the only variables entering are then the λ and the total action in terms of the nine variables $\{\lambda_a(b)\}$ assigned to the link a, ab is

$$\begin{aligned} S = \int L &= \lambda_u^2(u)\lambda_v^2(u) + \lambda_u^2(u)\lambda_v^2(v) + \lambda_u^2(v)\lambda_v^2(w) + \lambda_u^2(w)\lambda_v^2(u) + \lambda_u^2(w)\lambda_v^2(w) \\ &\quad + \lambda_u^2(v)\lambda_v^2(v) - 2\lambda_u(u)\lambda_v(v)\lambda_u(v)\lambda_w(u) \\ &\quad - 2\lambda_u(v)\lambda_v(w)\lambda_u(w)\lambda_w(v) - 2\lambda_u(w)\lambda_v(u)\lambda_u(u)\lambda_w(w) \end{aligned} \tag{21}$$

plus the cyclic rotations $u \to v \to w \to v$ of all terms. The first set of terms (which are $\int L_0$) can be written as a symmetric quadratic form D on the vector $(\lambda_u^2(u), \lambda_u^2(v), \lambda_u^2(w), \lambda_v^2(u), \ldots, \lambda_w^2(w))$. Here

$$D = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{22}$$

is diagnonalisable over $\mathbb{R}$ and has four eigenvectors of eigenvalue -1 and four of eigenvalue of 1/2. There is a final mode of eigenvalue 2 which is the vector with all entries $\lambda = 1$, which corresponds to $A = 0$. It corresponds to an equal length for all allowed directions. Because all the eigenmodes are real, we can linearise the theory about this configuration and our positivity constraints are not affected. On the other hand, the $\lambda = 1$ solution is an absolute minimum of the total action S (using Theorem 3.2). Hence all these fluctuations increase the energy of the configuration. In particular, we do not appear to have ‘metric waves’ in the theory for this model. Theorem 3.1 tells us that there are other 3-manifolds of flat connections in families (i), (ii) which are singular in the sense that some $\lambda_a(b)$ vanish. Their fluctuations (by the theorem) have three modes which keep the action zero but for which the connection remains singular, while other fluctuations increase the action. It appears from this discussion (without actually trying to do the integrals) that the ‘quantum statistical mechanics’ of this theory (i.e. integrals over the nine λ variables with weighting e^{-S}) has $\langle\lambda^a(b)\rangle > 0$. Note also that this ‘metric’ theory of these λ should not, however, be confused with the actual noncommutative Riemannian geometry as in [3] which is based on spin connections rather than $U(1)$ connections A, but it gives some flavour of the full theory.

4. Quantum electromagnetism

In this section we conclude with some basic aspects of the formulation of the quantum theory using a path integral approach. We will show that the quantum theory is fully computable. Indeed, functional integration in our setting becomes finite-dimensional iterated integrals and one can in fact do these integrals. For the present, we also omit physical constants and factors of i in the action since these are matter of taste. Since there is no preferred time direction one might think that the Euclidean theory is more appropriate.

We begin with the simplest case, a free scalar field

$$Z_A = \int D\phi\, e^{\int (d\phi)^* \wedge \star d\phi + V(\phi) + \int A^* \wedge \star J(\phi)}$$

for some potential $V(\phi)$ and possible coupling to an external field A. On shell, the current J is conserved but one should not exactly think that A is a Maxwell field. For a more geometrical theory of a particle moving in a background potential one should use the gauge theory and minimal coupling method (see below). The main feature of the above is that it is fully computable by elementary means, depending on the potential and external field. Of course there is nothing stopping from one doing some of these functional integrals (and those below) using Feynman diagram methods and a perturbative approach, which may be useful (depending on the potential V).

Equally elementary, we can quantise the Maxwell field with a classical external source J. Thus,

$$Z[J] = \int DA\, \mathrm{e}^{\int (\mathrm{d}A)^* \wedge \star(\mathrm{d}A) + A^* \wedge \star J}$$

where we have an infinite gauge degeneracy. This can be handled in several ways. For example, we regularise integrals to a finite volume of field strength of modulus $< \Lambda$, and take Λ to infinity. Gauge symmetry means a factor Λ^6 but in the ratios involved in vacuum expectation values this cancels, i.e. we can regulate and remove the regulator in all ratios with ease. More geometrically, we have already seen that the strengthened Coulomb gauge (5) in Section 2.4 is a complete gauge fixing. Hence we can impose these by integrating over a functional Lagrange multiplier field (Faddeev–Popov ghosts) for the $\sum_a \partial^a A^a = 0$ condition, and an additional constant Lagrange multiplier for the global $\int \sum_a A^a = 0$ condition.

On the other hand, neither of these conventional formalities are needed in our finite case. This is because we know that the operator $\star\mathrm{d}\star\mathrm{d}$ in Section 2.4, while not symmetric, can be diagonalised via Gram–Schmidt to orthonormal eigenvectors e_i say, $i = 1, \dots, 12$ for the 12D space of nonzero eigenvalue. Being eigenvectors these are also in the image of the operator and can therefore be viewed either as strongly conserved sources J or gauge potentials A in the strong Coulomb gauge. We have seen in our case that there are four eigenvectors each of eigenvalue $-3, -6, -9$. Clearly, if we write $A = \alpha^i e_i$ and $J = J^i e_i$ and the eigenvalues are λ_i then

$$Z[J] = \int \mathrm{d}^{12}\alpha^i\, \mathrm{e}^{2\lambda_i |\alpha_i|^2 + 2\bar{\alpha}_i J^i}.$$

We need here that $A^* \wedge \star J = A^{*a} e_a \wedge J^b \star e_b = 2R_a(\bar{A}^a J^a)\mathrm{Top}$ so that its integral is the usual l^2 inner product on S_3.

For a less trivial theory one can also couple the two theories above, thus

$$L = (\mathrm{d}\phi)^* \wedge (\mathrm{d}\phi) + (\mathrm{d}A)^* \wedge \star(\mathrm{d}A) + A^* \wedge J(\phi).$$

This is not gauge-invariant (except when ϕ is on shell) but it can still be functionally integrated over.

Finally, and more interesting than the essentially linear or Maxwell theory is the fully nonlinear Yang–Mills theory even in the $U(1)$ case. Here we have been rather more careful to impose the unitarity condition (because it has more of an impact) in our treatment in Section 3. In particular, we really do not need to gauge fix since the $U(1)^6$ symmetry gives a finite volume $(2\pi)^6$. Similarly, in this case there is a covariant derivative under a gauge symmetry $\phi \mapsto U\phi$ for charged scalar fields, e.g.

$$D_A\phi = (\mathrm{d} + A)\phi, \qquad D_A\phi \mapsto \mathrm{d}(U\phi) + (UAU^{-1} + U\,\mathrm{d}U^{-1})U\phi = UD_A\phi.$$

Then

$$L = F^* \wedge \star F + (D_A\phi)^* \wedge \star D_A\phi + V(\bar{\phi}\phi)$$

is the Lagrangian for the coupled system with some potential V. We have used part of this for the source $J(\phi)$ and this is its proper context. Of particular interest is the pure Yang–Mills theory. In lattice gauge theory, even for $U(1)$ one expects confinement as an artefact of the lattice regularisation. In our noncommutative geometrical version of lattice theory this appears as the $A \wedge A$ term which does not vanish precisely because the differential calculus is noncommutative. Thus it enters in the same 'form' as in non-Abelian gauge theory but for a different reason, but one may logically expect similar behaviour. Here we only want to note that our elementary 'Wilson loops' are in fact gauge-invariant and our result in Theorem 3.2 for the form of their action makes it particularly easy to compute them as follows. We define

$$Z[\mu_u, \mu_v, \mu_w] = \int \mathrm{d}A\, \mathrm{e}^{\int L_0 - \mu_u W_u - \mu_v W_v - \mu_w W_w},$$

where L_0 is the λ_a part of the Lagrangian given in (19). We can then compute the expectation values of elementary Wilson loops as

$$\langle W_a(x)\rangle = -Z^{-1} \left.\frac{\delta}{\delta\mu_a(x)}\right|_{\mu_u=\mu_v=\mu_w=1} (Z).$$

This is a matter of nine complex or 18 real integrals for the fields $\Phi^a(b)$ which determine the gauge configuration $A = \Phi - \theta$ (as explained in the proof of Theorem 3.1). We compute the detailed form of the theory now (actual numerical computations will be attempted elsewhere).

Thus, given the nine $\Phi^a(b)$ for $a, b \in \mathbb{C}$, the other $\Phi^a(x)$ are determined by the reality conditions, so we have only to integrates over all the possibles complex values for these nine. Next we adopt polar coordinates as in Theorem 3.2

$$\Phi^a(b) = \lambda_a(b)\,\mathrm{e}^{\mathrm{i}\theta^a(b)}, \quad \lambda_a(b) \in [0, \infty), \quad \theta^a(b) \in [0, 2\pi).$$

Including the Jacobian determinant, the partition function becomes

$$Z = 2^{-9} \int_0^\infty \mathrm{d}^9\lambda^2\, \mathrm{e}^{\int L_0(\lambda)} \int_0^{2\pi} \mathrm{d}^9\theta\, \mathrm{e}^{-\int W_u + W_v + W_w}$$

Here $\mathrm{d}^9\lambda^2 = \mathrm{d}\lambda_u^2(u) \cdots \mathrm{d}\lambda_w^2(w)$ as in Section 3.2 and $\mathrm{d}^9\theta = \mathrm{d}\theta^u(u) \cdots \mathrm{d}\theta^w(w)$. We omit the μ for simplicity. Next we write the Lagrangian in this integral explicitly in terms of these variables. Thus

$$\begin{aligned}\frac{1}{2}\int W_u + W_v + W_w = {}& \lambda^u(u)\lambda^v(v)\lambda^u(v)\lambda^w(u)\cos(\theta^u(u) - \theta^v(v) + \theta^u(v) \\ & - \theta^w(u)) + \lambda^u(v)\lambda^v(w)\lambda^u(w)\lambda^w(v)\cos(\theta^u(v) - \theta^v(w) \\ & + \theta^u(w) - \theta^w(v)) + \lambda^u(w)\lambda^v(u)\lambda^u(u)\lambda^w(w)\cos(\theta^u(w) \\ & - \theta^v(u) + \theta^u(u) - \theta^w(w))\end{aligned}$$

plus the cyclic rotations $u \to v \to w \to u$.

We concentrate on the θ-integrals, i.e. we write

$$Z = \int_0^\infty \mathrm{d}^9\lambda^2\, \mathrm{e}^{\int L_0(\lambda)} Z_\lambda,$$

where Z_λ is the partition function for the $U(1)$ lattice gauge theory defined by the θ variables with the λ variables held fixed. Next, gauge symmetry means that the Lagrangian here does not in fact depend on all nine of the θ parameters. In fact, it depends on only four, which can be made manifest by the transformation matrix:

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Explicitly, we replace the $\{\theta^a(b)\}$ by

$$\theta_1 = \theta^u(v) - \theta^v(w) + \theta^u(w) - \theta^w(v), \qquad \theta_2 = \theta^v(u) - \theta^w(v) + \theta^v(v) - \theta^u(u),$$
$$\theta_3 = \theta^v(w) - \theta^w(u) + \theta^v(u) - \theta^u(w), \qquad \theta_4 = \theta^w(v) - \theta^u(w) + \theta^w(w) - \theta^v(v)$$

and the remaining five

$$\theta_5 = \theta^v(v), \qquad \theta_6 = \theta^v(w), \qquad \theta_7 = \theta^w(u), \qquad \theta_8 = \theta^w(v), \qquad \theta_9 = \theta^w(w)$$

are unchanged. The determinant for this change of variables is 1. We also write

$$\lambda_1 = \lambda_u(u)\lambda_v(v)\lambda_u(v)\lambda_w(u), \qquad \lambda_2 = \lambda_u(v)\lambda_v(w)\lambda_u(w)\lambda_w(v),$$
$$\lambda_3 = \lambda_u(w)\lambda_v(u)\lambda_u(u)\lambda_w(w)$$

for the λ-holonomy expressions as in (21). Similarly

$$\begin{aligned}
&\lambda_4 = \lambda_v(u)\lambda_w(v)\lambda_v(v)\lambda_u(u), &&\lambda_5 = \lambda_v(v)\lambda_w(w)\lambda_v(w)\lambda_u(v),\\
&\lambda_6 = \lambda_v(w)\lambda_w(u)\lambda_v(u)\lambda_u(w), &&\lambda_7 = \lambda_w(u)\lambda_u(v)\lambda_w(v)\lambda_v(u),\\
&\lambda_8 = \lambda_w(v)\lambda_u(w)\lambda_w(w)\lambda_v(v), &&\lambda_9 = \lambda_w(w)\lambda_u(u)\lambda_w(u)\lambda_v(w)
\end{aligned}$$

for their cyclic rotations. Then we arrive at our final result

$$Z_\lambda = \int_0^{2\pi} \mathrm{d}\theta_5 \cdots \mathrm{d}\theta_9 \int_D \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_4 \, \mathrm{e}^{-S_\lambda(\theta_1,\theta_2,\theta_3,\theta_4)}, \tag{23}$$

where

$$\begin{aligned}
\tfrac{1}{2}S_\lambda = {}& \lambda_1 \cos(\theta_1 - \theta_2 + \theta_3) + \lambda_2 \cos(\theta_1) + \lambda_3 \cos(-\theta_2 - \theta_4) + \lambda_4 \cos(\theta_2)\\
&+ \lambda_5 \cos(-\theta_1 - \theta_4) + \lambda_6 \cos(\theta_3)\\
&+ \lambda_7 \cos(-\theta_1 - \theta_3) + \lambda_8 \cos(\theta_4) + \lambda_9 \cos(\theta_2 - \theta_3 + \theta_4)
\end{aligned}$$

and where the domain D is an affine transformation in $\mathbb{R}^4$ of the hypercube $[0, 2\pi)^4$, i.e. it has the form

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\end{pmatrix} = M([0, 2\pi)^4) + \begin{pmatrix}c_1\\ c_2\\ c_3\\ c_4\end{pmatrix},$$

where the linear transformation of the hypercube is given by

$$M = \begin{pmatrix} 0 & 1 & 1 & 0\\ -1 & 0 & 0 & 1\\ 0 & 0 & -1 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{24}$$

and the offsets (which are the only parts that depend on $\theta_5, \dots, \theta_9$) are

$$c_1 = -\theta_6 - \theta_8, \qquad c_2 = \theta_5 - \theta_8, \qquad c_3 = \theta_6 - \theta_7, \qquad c_4 = -\theta_5 + \theta_8 + \theta_9. \tag{25}$$

Clearly, one may compute the domain of integration $M([0, 2\pi)^4)$ for the variables $\theta_i' = \theta_i - c_i$ and thereby do the four θ' integrations followed by more trivial $\theta_5, \cdots, \theta_9$ integrals. Without doing the actual integrals, it is clear at this point that one obtains here some form of Bessel function (if we put an i in the action) as $\int_0^{2\pi} \mathrm{d}\theta \, \mathrm{e}^{\mathrm{i}\lambda\cos\theta} = 2\pi J_0(\lambda)$. Similarly higher Bessel functions for expectation values of the $U(1)$ Wilson loops $w_u(x)$, etc. This is a similar situation as conventional lattice gauge theory. In addition, we have the 'metric' λ integrals in our theory as discussed in Section 3.2.

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